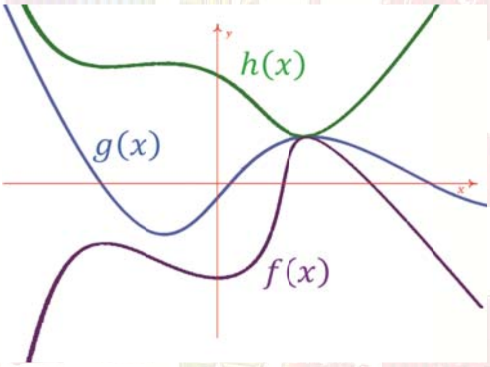
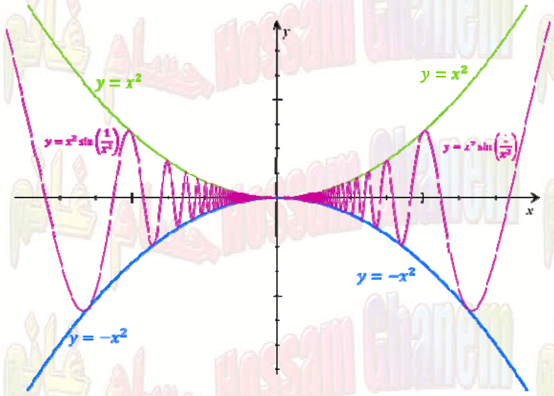
HOSSAM GHANEM

(6) 2.3 *The squeeze (Sandwich) Theorem*





The squeeze (Sandwich) Theorem

If $f(x)$, $g(x)$ and $h(x)$ are continuous functions

Such that $f(x) \leq g(x) \leq h(x)$

and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$

Then $\lim\limits_{x \to a} g(x) = L$

مسائل نهايات تحتوي على

$\sin\theta$ & $\cos\theta$

$\theta \to \infty$

نستخدم

The squeeze (Sandwich) Theorem

$\theta \to 0$

نستخدم القانون

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Example 1 59 9 July 2011	[2 pts.]: Let f be a function satisfying $4x - 9 \leq f(x) + x \leq x^2 - 4x + 7$, for all x in $(-\infty , \infty)$ Find $\lim_{x \to 4} f(x)$ (if it exists).

Solution

$L_1 = \lim_{x \to 4} 4x - 9 = 16 - 9 = 7$

$L_2 = \lim_{x \to 4} x^2 - 4x + 7 = 16 - 16 + 7 = 7$

$L_1 = L_2$

$\lim_{x \to 4} f(x) + x = 7$ by ST

$\lim_{x \to 4} f(x) + \lim_{x \to 4} x = 7$

$\lim_{x \to 4} f(x) + 4 = 7$

$\lim_{x \to 4} f(x) = 3$

Example 2 52 April 9, 2009 A	Find the following limit , if it exists $\lim_{x \to 1} (x-1)^{\frac{2}{3}} \cos\left(\frac{1}{x-1}\right)$

Solution

$L = \lim_{x \to 2} (x-1)^{\frac{2}{3}} \cos\left(\frac{1}{x-1}\right)$

$$-1 \leq \cos\left(\frac{1}{x-1}\right) \leq 1$$

$$(x-1)^{\frac{2}{3}} \geq 0$$

$$-(x-1)^{\frac{2}{3}} \leq (x-1)^{\frac{2}{3}} \cos\left(\frac{1}{x-1}\right) \leq (x-1)^{\frac{2}{3}} \quad \rightarrow (1)$$

$\lim_{x \to 1} -(x-1)^{\frac{2}{3}} = 0 \quad \rightarrow (2)$

$\lim_{x \to 1} (x-1)^{\frac{2}{3}} = 0 \quad \rightarrow (3)$

from (1), (2), (3) by ST

$\therefore \lim_{x \to 1} (x-1)^{\frac{2}{3}} \cos\left(\frac{1}{x-1}\right) = 0$

Example 3 49 July 5, 2008	Evaluate the following limit	$\lim_{\theta\to 0}\dfrac{\theta^2}{2+\cos\left(\frac{1}{\theta}\right)}$

Solution

$$L = \lim_{\theta\to 0}\frac{\theta^2}{2+\cos\left(\frac{1}{\theta}\right)}$$

$$-1 \le \cos\left(\frac{1}{\theta}\right) \le 1$$

$$1 \le 2+\cos\left(\frac{1}{\theta}\right) \le 3$$

$$\frac{1}{3} \le \frac{1}{2+\cos\left(\frac{1}{\theta}\right)} \le 1$$

$$\theta^2 \ge 0$$

$$\frac{\theta^2}{3} \le - \frac{\theta^2}{2+\cos\left(\frac{1}{\theta}\right)} \le \theta^2 \qquad \to (1)$$

$$\lim_{\theta\to 0}\frac{\theta^2}{3} = 0 \qquad \to (2)$$

$$\lim_{\theta\to 0}\theta^2 = 0 \qquad \to (3)$$

from (1), (2), (3) by ST

$$\therefore \lim_{\theta\to 0}\frac{\theta^2}{2+\cos\left(\frac{1}{\theta}\right)} = 0$$

Example 4 34 March 23,2002	Find the following limit , if it exists	$\lim_{x\to 0}\left(3+5\lvert x\rvert\cos\left(\frac{2}{x}\right)\right)$

Solution

$$L = \lim_{x\to 0}\left(3+5|x|\cos\left(\frac{2}{x}\right)\right)$$

$$-1 \le \cos\left(\frac{2}{x}\right) \le 1$$

$$|x| \ge 0$$

$$-5|x| \le 5|x|\cos\left(\frac{2}{x}\right) \le 5|x|$$

$$3-5|x| \le 3+5|x|\cos\left(\frac{2}{x}\right) \le 3+5|x| \qquad \to (1)$$

$$\lim_{x\to 0^-}(3-5|x|) = \lim_{x\to 0}(3+5x) = 3$$

$$\lim_{x\to 0^+}(3-5|x|) = \lim_{x\to 0}(3-5x) = 3$$

$$\therefore \lim_{x\to 0}(3-5|x|) = 3 \qquad \to (2)$$

also $\lim_{x\to 0}(3+5|x|) = 3 \qquad \to (3)$

from (1), (2), (3) by ST

$$\therefore \lim_{x\to 0}\left[3+5|x|\cos\left(\frac{2}{x}\right)\right] = 3$$

Example 5 38 March 31, 2004	Evaluate the following limit $\lim_{x\to 3}\left[3x+(x-3)^4\sin\frac{1}{\sqrt[3]{x-3}}\right]$

Solution

$L=\lim_{x\to 0}\left[3x+(x-3)^4\sin\frac{1}{\sqrt[3]{x-3}}\right]$

$$-1 \le \sin\frac{1}{\sqrt[3]{x-3}} \le 1$$

$$(x-3)^4 \ge 0$$

$$-(x-3)^4 \le (x-3)^4\sin\frac{1}{\sqrt[3]{x-3}} \le (x-3)^4$$

$$3x-(x-3)^4 \le 3x+(x-3)^4\sin\frac{1}{\sqrt[3]{x-3}} \le 3x+(x-3)^4 \qquad \to (1)$$

$\lim_{x\to 3} 3x-(x-3)^4 = 9 \qquad \to (2)$

$\lim_{x\to 3} 3x+(x-3)^4 = 9 \qquad \to (3)$

from (1), (2), (3) by ST

$$\therefore \lim_{x\to 3}\left[3x+(x-3)^4\sin\frac{1}{\sqrt[3]{x-3}}\right]=9$$

Example 6 45 March 28, 2007	Find the limit , if it exists $\lim_{x\to 0}\left(\sqrt{1+x^2}+\frac{x^2}{\sec\left(\frac{1}{x}\right)}\right)$

Solution

$L=\lim_{x\to 0}\left(\sqrt{1+x^2}+\frac{x^2}{\sec\left(\frac{1}{x}\right)}\right)=\lim_{x\to 0}\left(\sqrt{1+x^2}+x^2\cos\left(\frac{1}{x}\right)\right)$

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

$$x^2 > 0$$

$$-x^2 \le x^2\cos\left(\frac{1}{x}\right) \le x^2$$

$$\sqrt{1+x^2}-x^2 \le \sqrt{1+x^2}+x^2\cos\left(\frac{1}{x}\right) \le \sqrt{1+x^2}+x^2 \qquad \to (1)$$

$\lim_{x\to 0} \sqrt{1+x^2}-x^2 = 1 \qquad \to (2)$

$\lim_{x\to 0} \sqrt{1+x^2}+x^2 = 1 \qquad \to (3)$

from (1), (2), (3) by ST

$\therefore L=1$

Example 7 48 March 25, 2008 A	Find the limit , if it exists $\lim_{x\to 1}\left[x^5+(x^2-2x+1)\sin\left(\frac{1}{x-1}\right)\right]$

Solution

$$L = \lim_{x\to 1}\left[x^5+(x^2-2x+1)\sin\left(\frac{1}{x-1}\right)\right]$$

$$-1 \le \sin\left(\frac{1}{x-1}\right) \le 1$$

$$(x-1)^2 \ge 0$$

$$-(x-1)^2 \le (x-1)^2 \sin\left(\frac{1}{x-1}\right) \le (x-1)^2$$

$$x^5-(x-1)^2 \le x^5+(x-1)^2 \sin\left(\frac{1}{x-1}\right) \le x^5+(x-1)^2$$

$$x^5-(x-1)^2 \le x^5+(x^2-2x+1) \sin\left(\frac{1}{x-1}\right) \le x^5+(x-1)^2 \qquad \rightarrow (1)$$

$\lim_{x\to 1}[x^5-(x-1)^2] = 1 \qquad \rightarrow (2)$

$\lim_{x\to 1}[(x-1)^2+x^5] = 1 \qquad \rightarrow (3)$

from (1), (2), (3) by ST

$\therefore L = \lim_{x\to 1}\left[x^2+(x^2-2x+1)\sin\left(\frac{1}{x-1}\right)\right] = 1$

Example 8 36 April 19,2003 A	Find the following limit , if it exists $\lim_{x\to 0}\frac{\sin x + x^3\sin\frac{1}{x}}{x}$

Solution

$L = \lim_{x\to 0}\frac{\sin x + x^3\sin\frac{1}{x}}{x} = \lim_{x\to 0}\frac{\sin x}{x} + x^2\sin\frac{1}{x}$

$L_1 = \lim_{x\to 0}\frac{\sin x}{x} = 1$

$L_2 = \lim_{x\to 0} x^2\sin\frac{1}{x}$

$$-1 \le \sin\frac{1}{x} \le 1$$

$$x^2 \ge 0$$

$$-x^2 \le x^2\sin\frac{1}{x} \le x^2 \qquad \rightarrow (1)$$

$\lim_{x\to 0} -x^2 = 0 \qquad \rightarrow (2)$

$\lim_{x\to 0} x^2 = 0 \qquad \rightarrow (3)$

from (1), (2), (3) by ST

$\therefore L_2 = \lim_{x\to 0} x^2\sin\frac{1}{x} = 0$

$L = L_1 + L_2 = 1 + 0 = 1$

Homework

1	Find the limit , if it exists	$\lim_{x\to 2}\lvert x-2\rvert\cos^2\left(\frac{1}{x-2}\right)$	46 Date: July 5, 2007
2	Evaluate the limit(if it exists)	$\lim_{x\to 0} x^4\sin\left(\frac{1}{\sqrt[3]{x}}\right)$	25 January 12 .2003
3	Evaluate the limit (if it exists)	$\lim_{x\to 3} (x-3)^2\cos\left(\frac{1}{x-3}\right)$	10 October 27, 1994
4	Find	$\lim_{x\to 1} (x-1)^2\sin\left(\frac{1}{x-1}\right)$	30 October 19, 2000 A 8 October 28, 1993
5	Find the limit , if it exists	$\lim_{x\to 0} x^{\frac{2}{3}}\sin\left(\frac{1}{x}\right)$	12 November 2, 1995
6	Evaluate the limit(if it exists)	$\lim_{x\to 0}\lvert x\rvert\cos\left(\frac{\pi}{x}\right)$	32 March 22, 2001
7	Find the limit , if it exists	$\lim_{x\to 1}(x-1)^2\cos\left(\frac{1}{x-1}\right)$	29 Feb 24, 2000 41 March 30, 2005
8	Evaluate the limit(if it exists)	$\lim_{x\to\infty}\frac{1}{x^2}\sin(x-1)$	53 July 18, 2009 A
9	Evaluate the limit (if it exists)	$\lim_{x\to 0}\left(1+x^2\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right)$	7 July 29, 1993
10	Find	$\lim_{x\to 0}\frac{x^2}{\sqrt{x^4+x^2+8}}$	
11	Find the limit , if it exists	$\lim_{x\to 0} x^2\sin\left(\frac{1}{\sqrt[3]{x}}\right)$	4 May 19, 1992

Homework

12	Find the limit , if it exists $\lim_{x-1}\left[(x-1)^2\sin\left(\frac{\pi}{x-1}\right)+2\right]$	40 October 28, 2004 A
13	Find the limit , if it exists $\lim_{x\to 2}\ \lvert x-2\rvert\ \sin\left(\frac{\pi}{x-2}\right)$	33 October 25,2001 A
14	Let f and g be two function defined on $(-\infty\,,\infty\,)$ if $0\le 2g(x)-f(x)\le x^2$ for all x and $\lim_{x\to 0}f(x)=6$ Then find $\lim_{x\to 0}g(x)$	5 April 8, 1993
15	Let F be a function such that $a<f(x)<b$ for $x\in(-\infty\,,\infty\,)$ if $g(x)=7x^4+3x^2\sqrt{x^2+1}$ Find $\lim_{x\to 0}f(x)g(x)$	
16	Let $x-2\le f(x)\le \sin x-g(x)$if $g(x)$is continuous at $x=0$ and . $g(0)=2$ Then evaluate $\lim_{x\to 0}f(x)$	
17	If $f(x)$ satisfies the inequality $(3-\lvert x-4\rvert)\le f(x)\le(x-1)$ Find $\lim_{x\to 3}x^2f(x)$	11 March 31, 1994
18	Find $\lim_{x\to 2}\left[\frac{6x^2-11x-2}{x^3-8}+(x-2)^2\sin\frac{1}{(x-2)^2}\right]$	30 Jan. 12. 2008
19	Find the following limit , if it exists $\lim_{x\to 0}\frac{\tan(2x)+x^3\sin\left(\frac{4}{x}\right)}{x}$	28 January 13. 2007
20	Compute the following limits if they exist. If a limit does not exist clearly state why. $\lim_{x\to\infty}\frac{x\sin x}{x^2+1}$	58 7April 2011

Homework

21	Assume $\lim_{x\to 1} f(x)$ exists and $$1 \le \frac{f(x)}{(x+1)^2} \le \frac{x^3-1}{3(x-1)}$$ Find $\lim_{x\to 1} f(x)$. (3 points)	56 July 10, 2010
22	(2pts) Evaluate the following limits, if they exist. $\lim_{x\to 0} \frac{x}{2+\sin\left(\frac{1}{x}\right)}$	57 November 8, 2010
23	Evaluate the following limits, if they exist: $\lim_{x\to 0} x^{\frac{2}{3}} \sin\frac{1}{x}$	38 January 15, 2011
24	Assume $\lim_{x\to -1} f(x)$ exists and $$\frac{x^2+x-2}{x+3} \le \frac{f(x)}{(x-1)^2} \le \frac{x^2+2x-1}{x+3}$$ find $\lim_{x\to -1} f(x)$	
25	Evaluate the following limits, if they exist: $\lim_{x\to 0} \left[(2x+1) + x^4 \cos\left(\frac{1}{x^4}\right)\right]$	40 August 7, 2011
26	Let $x-5 \le f(x) \le \sin x - g(x)$ if $g(x)$ is continuous at $x=0$ and $g(0)=5$ Then evaluate $\lim_{x\to 0} f(x)$	
27	Find the limit , if it exists $\lim_{x\to 2} (x-2)^2 \sin\left(\frac{1}{x-2}\right)$	14 March 28, 1996
28	Find the limit , if it exists $\lim_{x\to 1} \frac{(x-1)^2}{\sqrt{x^4+2x^2+8}}$	9 November 1993
29	[2 Pts.] Evaluate the following limit , if it exist $\lim_{x\to 2} (x-2)^2 \cos\left(\frac{2}{x-2}\right)$	41 7 January 2012

24	Assume $\lim\limits_{x\to-1} f(x)$ exists and $\dfrac{x^2+x-2}{x+3} \le \dfrac{f(x)}{(x-1)^2} \le \dfrac{x^2+2x-1}{x+3}$ find $\lim\limits_{x\to-1} f(x)$

Solution

$\lim\limits_{x\to-1} \dfrac{x^2+x-2}{x+3} = \dfrac{1-1-2}{-1+3} = \dfrac{-2}{2} = -1 \qquad \to (1)$

$\lim\limits_{x\to-1} \dfrac{x^2+2x-1}{x+3} = \dfrac{1-2-1}{-1+3} = \dfrac{-2}{2} = -1 \qquad \to (2)$

but $\dfrac{x^2+x-2}{x+3} \le \dfrac{f(x)}{(x-1)^2} \le \dfrac{x^2+2x-1}{x+3} \qquad \to (3)$

from (1), (2), (3) by ST

$\therefore \lim\limits_{x\to-1} \dfrac{f(x)}{(x-1)^2} = -1$

$\because \lim\limits_{x\to-1} f(x)$ exists

$\therefore \lim\limits_{x\to-1} \dfrac{f(x)}{(x-1)^2} = \dfrac{\lim\limits_{x\to-1} f(x)}{\lim\limits_{x\to-1} (x-1)^2}$

$\therefore \dfrac{\lim\limits_{x\to-1} f(x)}{\lim\limits_{x\to-1} (x-1)^2} = -1$

$\therefore \lim\limits_{x\to-1} f(x) = -\lim\limits_{x\to-1} (x-1)^2 = -(-1-1)^2 = -4$

26	Let $x-5 \le f(x) \le \sin x - g(x)$ if $g(x)$ is continuous at $x=0$ and $g(0)=5$ Then evaluate $\lim\limits_{x\to0} f(x)$

Solution

$\because g(x)$ cont.

$\therefore \lim\limits_{x\to0} g(x) = g(0) = 5$

$\lim\limits_{x\to0} \sin x - g(x) = 0-5 = -5 \qquad \to (1)$

$\lim\limits_{x\to0} (x-5) = -5 \qquad \to (2)$

$\because x-5 \le f(x) \le \sin x - g(x) \qquad \to (3)$

from (1), (2), (3) by S.T

$\therefore \lim\limits_{x\to0} f(x) = -5$

27 14 March 28, 1996	Find the limit , if it exists $\lim\limits_{x\to 2} (x-2)^2 \sin\left(\frac{1}{x-2}\right)$

Solution

$$L = \lim_{x\to 2} (x-2)^2 \sin\left(\frac{1}{x-2}\right)$$

$$-1 \le \sin\left(\frac{1}{x-2}\right) \le 1$$

$$(x-2)^2 \ge 0$$

$$-(x-2)^2 \le (x-2)^2 \sin\left(\frac{1}{x-2}\right) \le (x-2)^2 \quad \to (1)$$

$$\lim_{x\to 2} -(x-2)^2 = 0 \quad \to (2)$$

$$\lim_{x\to 2} (x-2)^2 = 0 \quad \to (3)$$

from (1), (2), (3) by ST

$$\therefore \lim_{x\to 2} (x-2)^2 \sin\left(\frac{1}{x-2}\right) = 0 \qquad S.T$$

28 9 November 1993	Find the limit , if it exists $\lim\limits_{x\to 1} \frac{(x-1)^2}{\sqrt{x^4+2x^2+8}}$

Solution

$$L = \lim_{x\to 1} \frac{(x-1)^2}{\sqrt{x^4+2x^2+8}} = \frac{1-1}{\sqrt{1+2+8}} = 0$$